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The Scattering of an Electromagnetic Wave by a Semi-infinite Conducting Cylinder Capped by a Conducting Hemisphere in the Long Wavelength Limit

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LINCOLN LABORATORY

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THE SCATTERING OF AN ELECTROMAGNETIC WAVE BY A SEMI-INFINITE CONDUCTING CYLINDER CAPPED BY A CONDUCTING HEMISPHERE IN THE LONG WAVELENGTH LIMIT.

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ABSTRACT

We consider a scatterer which consists of a right circular semi-infinite conducting cylinder capped by a conducting hemisphere of the same radius as the cylinder. We take the positive z axis as the axis of the cylinder. Plane polarized electromagnetic waves whose direction of propation is the positive z axis and which come from negative infinity on the z axis are incident upon the scatterer. We require the differential cross section in the long wavelength limit. The cross section is obtained through the use of an approximate surface current.

It is found that the cross section varies as the inverse square of the wavelength instead of the inverse fourth power as in Rayleigh scattering.

I. INTRODUCTION AND SUMMARY

Let us consider a scatterer which consists of a semi-infinite right conducting cylinder to which there is attached a conducting hemisphere of the same radius at the end of the cylinder. The entire scatterer is thus smooth and everywhere convex.

Let us denote the radius of the cylinder and hemisphere by a. We choose a system of coordinates such that the positive z-axis coincides with the axis of the cylinder and such that the plane z=0 terminates the cylinder. The hemisphere is below the z=0 plane which also terminates the hemisphere from above. We wish to obtain the differential scattering cross section in the long wavelength limit. We consider a monochromatic electromagnetic wave whose wave number is k $(= \frac{2\pi}{\lambda})$, which is polarized in the x direction, and which approaches the scatterer from the negative z-direction as the incident wave.

The scattering problem w~~hich we pose~~ differs from the usual long wavelength limit scattering problem--called the "Rayleigh scattering problem"--in an essential aspect. In the present paper ~~we shall consid~~er the ~~case that the~~ wavelength λ is large compared to the radius a, that is ka is small compared to unity. Of course, since the cylinder is semi-infinite, λ is small compared to the length of the scatterer. In Rayleigh scattering, by contrast, all linear dimensions of the scatterer are small compared to the wavelength, (see e.g. ref. 1). It can be shown for Rayleigh scattering

lambda

that the scattering cross section varies as k^4. In fact for a sphere the echo area A_e is given by

$$A_e = 9\pi a^6 k^4$$
$$\simeq 2V^2 k^4 \quad , \tag{1.1}$$

where a and V are the radius and volume of the sphere. For smooth scatterers which vary not too much from a sphere in shape we may expect the approximate formula of (1.1) to hold. Exact results depend upon the possibility of solving certain potential problems exactly.

In the problem which we have set up we cannot use the techniques for Rayleigh scattering directly because of the infinite dimension. Since we cannot solve the exact electromagnetic problem, we use an approximate technique which is based upon the possibility of writing the scattering amplitude, from which the differential cross section is calculated, as an integral over the surface current. Since the current appears in an integral which extends over a surface of infinite extent, we assume that local errors in the current distribution will not affect the scattering amplitude very much, at least for order-of-magnitude calculations. We then make "reasonable" assumptions for the surface current distribution. For the hemisphere portion of the scatterer we

assume the same current distribution (in the long wavelength limit) as for the illuminated side of a sphere of the same radius, if the sphere were the scatterer. For the cylindrical portion of the scatterer we solve in the long wavelength limit an electromagnetic problem which we believe to hold over most of the cylinder. We then calculate the current on the cylinder, assuming that it is continuous at the junction of the cylinder and hemisphere.

It turns out that for back scattering the surface current on the cylinder makes no contribution. The result is identical to that obtained for a spherical scatterer if one ignores the current on the unilluminated side.

We shall now give our results. From our choice of coordinate system it is seen that the origin of coordinates coincides with the center of the sphere from which the hemisphere portion of the scatterer is taken. Let $\underset{\sim}{r}$ be the vector drawn from the origin to the observer. Let the polar coordinates of $\underset{\sim}{r}$ be given by (r,θ,φ) and the cartesian coordinates be given by (x,y,z) so that

$$\begin{aligned} x &= r \cos\varphi \sin\theta \ , \\ y &= r \sin\varphi \sin\theta \ , \\ z &= r \cos\theta \ . \end{aligned} \tag{1.2}$$

Let $\sigma(\theta,\varphi)$ be the differential scattering cross section. Then

$$I_o \, \sigma(\theta,\varphi) \sin\theta d\theta d\varphi$$

is the (time-averaged) energy scattered through the differential surface angle sin θ dθdφ in a direction specified by θ, φ where I_o is the (time-averaged) incident energy per unit area. Our result is

$$\sigma(\theta,\varphi) = \frac{9}{64} a^2 (ka)^2 \{\cos^2\varphi[\cos\theta - \pi A(\theta) \sin^2\theta]^2 + \sin^2\varphi\}, \quad (1.3)$$

where the function $A(\theta)$ is given by

$$\begin{aligned} A(\theta) &= 1 \quad , \quad 0 \le \theta < \frac{\pi}{2} \\ &= 0 \qquad \frac{\pi}{2} \le \theta \le \pi \quad . \end{aligned} \quad (1.3a)$$

The echo area A_e is defined by

$$A_e = 4\pi\sigma(\pi,\varphi). \quad (1.4)$$

Thus for our scatterer we have

$$A_e = \frac{9}{16} \pi a^2 (ka)^2 \quad , \quad (1.5)$$

which is the principal result of the present paper. The result (1.5) is of course quite different than the Rayleigh scattering result (1.1).

II. THE SCATTERING FORMALISM IN TERMS OF SURFACE CURRENTS

For a perfectly conducting scatterer it can be shown (reference 2) that the electric field is given by

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}^{inc}(\underset{\sim}{r}) + i\omega\mu \int_S \Gamma(\underset{\sim}{r},\underset{\sim}{r}')\underset{\sim}{J}(\underset{\sim}{r}')dS' \tag{2.1}$$

in terms of rationalized MKS units with the time factor taken as $e^{-i\omega t}$. In (2.1) $\underset{\sim}{r}$ is the field point, $\underset{\sim}{E}^{inc}$ is the incident electric field which we take to have the form

$$\underset{\sim}{E}^{inc}(\underset{\sim}{r}) = E_o e^{ikz} \underset{\sim}{i}_x \tag{2.2}$$

where $\underset{\sim}{i}_x$, $\underset{\sim}{i}_y$, $\underset{\sim}{i}_z$ are the unit vectors along the x,y,z coordinates respectively, the integral is taken over the surface S of the scatterer with $\underset{\sim}{r}'$ being a point on the surface of the scatterer, $\underset{\sim}{J}(\underset{\sim}{r})$ is the surface current and Γ is a matrix Green's function whose components are

$$\Gamma_{ij}(\underset{\sim}{r},\underset{\sim}{r}') = [\delta_{ij} + \frac{1}{k^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}] \frac{1}{4\pi} \frac{\exp[ik|\underset{\sim}{r}-\underset{\sim}{r}'|]}{|\underset{\sim}{r}-\underset{\sim}{r}'|} . \tag{2.3}$$

We take ϵ and μ to have their free space values.

The magnetic field $\underset{\sim}{H}$ is given by

$$\underset{\sim}{H} = -\frac{i}{\omega\mu} \underset{\sim}{\nabla} \times \underset{\sim}{E} . \tag{2.4}$$

The incident magnetic field is

$$\underset{\sim}{H}^{inc}(\underset{\sim}{r}) = \frac{E_o}{\eta} e^{ikz} \underset{\sim}{i}_x \quad , \tag{2.5}$$

where

$$\eta = [\mu/\varepsilon]^{1/2} \quad . \tag{2.6}$$

If we define I_o to be the incident (time-averaged) energy flux we have

$$\begin{aligned} I_o &= \sqrt{2}\, [\underset{\sim}{E}^{inc} \times \underset{\sim}{H}^{inc*}]_z \\ &= \sqrt{2}\, \frac{E_o^2}{\eta} \quad . \end{aligned} \tag{2.7}$$

On using the notation $\underset{\sim}{E}^{sc}(\underset{\sim}{r})$ for the scattered field we have (if $\underset{\sim}{J}$ dies down rapidly enough)

$$\underset{\sim}{E}^{sc}(\underset{\sim}{r}) = -i\frac{\omega\mu}{4\pi} \frac{e^{ikr}}{r} \underset{\sim}{\hat{r}} \times \underset{\sim}{\hat{r}} \times \int_S \underset{\sim}{J}(\underset{\sim}{r}')\exp[-ik(\underset{\sim}{\hat{r}}\cdot\underset{\sim}{r}')]dS' \quad , \tag{2.8}$$

where $\underset{\sim}{\hat{r}}$ is the unit vector in the direction of $\underset{\sim}{r}$. $\underset{\sim}{H}^{sc}$ is obtained from (2.4) on replacing $\underset{\sim}{E}$ by $\underset{\sim}{E}^{sc}$. The differential scattering cross section $\sigma(\theta,\varphi)$ is obtained from the expression

$$\sigma(\theta,\varphi) = \sqrt{2}\, \frac{[\underset{\sim}{E}^{sc} \times \underset{\sim}{H}^{sc*}]\cdot\underset{\sim}{\hat{r}}}{I_o} r^2 \quad . \tag{2.9}$$

In using (2.8) and (2.9) to evaluate the cross section it is convenient to split the integral which occurs in (2.8) into two parts: an integral over the spherical cap and an integral over the semi-infinite cylinder. Hence we shall define the vectors $\underset{\sim}{I}_1$ and $\underset{\sim}{I}_2$ by

$$\underset{\sim}{I}_j = \hat{\underset{\sim}{r}} \times \hat{\underset{\sim}{r}} \times \int_{S_j} \underset{\sim}{J}(\underset{\sim}{r}') \exp[-ik(\hat{\underset{\sim}{r}}\cdot\underset{\sim}{r}')]dS' \tag{2.8a}$$

where S_1 is the surface of the spherical cap and S_2 is the surface of the cylinder.

III. EVALUATION OF THE CONTRIBUTION OF THE SPHERICAL CAP.

We shall now evaluate $\underset{\sim}{I}_1$.

From Reference 3 we obtain the following expression for the surface currents on a perfectly conducting sphere which is illuminated by the incident wave (2.2):

$$J_\theta = \frac{E_o}{\eta}\frac{\cos\varphi}{ka}\sum_{n=1}^{\infty}\frac{i^{n-1}(2n+1)}{n(n+1)}\left[\frac{\sin\theta\, P_n^{1'}(\cos\theta)}{\hat{H}_n^{(1)'}(ka)} - i\,\frac{P_n^1(\cos\theta)}{\sin\theta\,\hat{H}_n^{(1)}(ka)}\right],$$

$$J_\varphi = \frac{E_o}{\eta}\frac{\sin\varphi}{ka}\sum_{n=1}^{\infty}\frac{i^{n-1}(2n+1)}{n(n+1)}\left[\frac{P_n^1(\cos\theta)}{\sin\theta\,\hat{H}_n^{(1)'}(ka)} - i\,\frac{\sin\theta\, P_n^{1'}(\cos\theta)}{\hat{H}_n^{(1)}(ka)}\right]$$

(3.1)

In (3.1) a is the radius of the sphere as usual while J_θ and J_φ are the components of the surface current vector in terms of polar coordinates. The prime indicates that the derivative of the function is to be used. The function $\hat{H}_n^{(1)}(x)$ is defined by

$$\hat{H}_n^{(1)}(x) = xh_n^{(1)}(x) \quad , \tag{3.2}$$

where $h_n^{(1)}$ is the usual spherical Bessel function. We should also note that we have taken the complex conjugate of the currents given in reference 3 because we are using the time factor $e^{-i\omega t}$ instead of $e^{i\omega t}$.

We now take ka to be much less that 1. For small x we note that

$$\hat{H}_n^{(1)}(x) \cong -i \frac{(2n)!}{2^n n! x^n} \quad , \tag{3.3}$$

$$\hat{H}_n^{(1)'}(x) \cong i \frac{(2n)!}{2^n (n-1)! x^{n+1}} \quad . \tag{3.4}$$

Thus for very small ka (3.1) gives us the following expressions for the components of the surface current

$$J_\theta = -\frac{3}{2}\frac{E_o}{\eta} \cos\varphi \quad ,$$

$$J_\varphi = \frac{3}{2}\frac{E_o}{\eta} \sin\varphi \cos\theta \quad . \tag{3.5}$$

In accordance with our discussion in the Introduction we shall use the current (3.5) in the expression for $\underset{\sim}{I}_1$.

To evaluate the integral $\underset{\sim}{I}_1$ we use $\underset{\sim}{J}(\underset{\sim}{r}')$ which appears in (2.8a) in the form

$$\underset{\sim}{J}(\underset{\sim}{r}') = J_\theta(\varphi')\,\hat{\underset{\sim}{\theta}}' + J\,(\theta',\varphi')\hat{\underset{\sim}{\varphi}}' \; , \tag{3.6}$$

where $\hat{\underset{\sim}{\theta}}'$ and $\hat{\underset{\sim}{\varphi}}'$ are unit vectors on the spherical cap at the point given by $\underset{\sim}{r}'$ in the directions given by the polar angles. It will be useful to introduce a fixed set of orthogonal unit vectors determined by the position of the observer $\underset{\sim}{r}$. We shall denote this set by $\hat{\underset{\sim}{r}}$, $\hat{\underset{\sim}{\theta}}$, $\hat{\underset{\sim}{\varphi}}$. In terms of this set of fixed vectors we have

$$\begin{aligned}
\hat{\underset{\sim}{\theta}}' = {} & [\sin\theta\cos\theta'\cos(\varphi'-\varphi) - \sin\theta'\cos\theta]\,\hat{\underset{\sim}{r}} \\
& + [\cos\theta\cos\theta'\cos(\varphi'-\varphi) + \sin\theta'\sin\theta]\,\hat{\underset{\sim}{\theta}} \\
& + [\cos\theta'\sin(\varphi'-\varphi)]\,\hat{\underset{\sim}{\varphi}} \quad ,
\end{aligned}$$

$$\begin{aligned}
\hat{\underset{\sim}{\varphi}}' = {} & [-\sin\theta\sin(\varphi'-\varphi)]\,\hat{\underset{\sim}{r}} \\
& + [-\cos\theta\sin(\varphi'-\varphi)]\,\hat{\underset{\sim}{\theta}} \\
& + [\cos\,(\varphi'-\varphi)]\,\hat{\underset{\sim}{\varphi}}
\end{aligned} \tag{3.7}$$

where the unprimed angles are th polar angles associated with $\underset{\sim}{\hat{r}}$ and the primed angles are associated with $\underset{\sim}{\hat{r}}'$.

We now use (3.7) in (3.6) and substitute the result into (2.8a). In addition, we replace the exponential by unity, since ka is small. The integration is now easily carried out and we obtain

$$\underset{\sim}{I}_1 = \frac{3\pi}{2} \frac{E_o}{\eta} a^2[\underset{\sim}{\hat{\theta}} \cos\theta \cos\varphi - \underset{\sim}{\hat{\varphi}} \sin\varphi]. \tag{3.8}$$

IV. THE EVALUATION OF THE CONTRIBUTION ON THE CYLINDER

We now wish to evaluate $\underset{\sim}{I}_2$. We shall need the surface current on the cylindrical portion of the scatterer. We shall obtain this current by solving an approximate boundary value problem for $\underset{\sim}{E}$ and $\underset{\sim}{H}$ for the cylindrical portion. The surface cureent will then be given by

$$\underset{\sim}{J} = \underset{\sim}{n} \times \underset{\sim}{H} \tag{4.1}$$

where $\underset{\sim}{n}$ is the normal to the surface of the cylinder and $\underset{\sim}{H}$ is evaluated at the surface. It will turn out that the surface current, as evaluated in this manner, will contain two arbitrary constants which arise from the fact that we have not completely specified the solution of the boundary value problem. These constants are obtained by the requirement that the surface current be continuous at the junction of the spherical cap and the cylinder.

We shall use the techniques of Reference 3 (Chapter 5) to obtain general solutions of Maxwell's Equations with cylindrical symmetry. We shall then put increasingly severe requirements on these solutions so that they are "reasonable" solutions from a physical point of view in the sense that at a distance from the junction of the spherical cap to the cylinder that they are waves propagating in essentially the positive z direction along a thin wire. In order to make it easier to refer to Reference 3 we shall use as a time factor $e^{i\omega t}$ as in Reference 3 for the time being instead of $e^{-i\omega t}$ as in Reference 2 and in the previous sections of this report.

In accordance with Reference 3 we introduce two scalar potential functions $\psi_1(\underset{\sim}{r})$, $\psi_2(\underset{\sim}{r})$, each of which satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\psi_i = 0 . \qquad (4.2)$$

Every electromagnetic field $\underset{\sim}{E}(\underset{\sim}{r})$, $\underset{\sim}{H}(\underset{\sim}{r})$ can be split up as

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}^{(1)}(\underset{\sim}{r}) + \underset{\sim}{E}^{(2)}(\underset{\sim}{r}),$$

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}^{(1)}(\underset{\sim}{r}) + \underset{\sim}{H}^{(2)}(\underset{\sim}{r}) \qquad (4.3)$$

such that the components of the vectors $\underset{\sim}{E}^{(i)}$, $\underset{\sim}{H}^{(i)}$ in cylindrical coordinates are given by

$$E_\rho^{(1)} = -\frac{i}{\omega\epsilon}\frac{\partial^2\psi_1}{\partial\rho\partial z}, \qquad H_\rho^{(1)} = \frac{1}{\rho}\frac{\partial\psi_1}{\partial\varphi}$$

$$E_\varphi^{(1)} = -\frac{i}{\omega\epsilon\rho}\frac{\partial^2\psi_1}{\partial\varphi\partial z}, \qquad H_\varphi^{(1)} = -\frac{\partial\psi_1}{\partial\rho}$$

$$E_z^{(1)} = -\frac{1}{\omega\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1, \qquad H_z^{(1)} = 0 \quad .$$

$$E_\rho^{(2)} = -\frac{1}{\rho}\frac{\partial\psi_2}{\partial\varphi}, \qquad H_\rho^{(2)} = -\frac{i}{\omega\mu}\frac{\partial^2\psi_2}{\partial\rho\partial z}$$

$$E_\varphi^{(2)} = \frac{\partial\psi_2}{\partial\rho}, \qquad H_\varphi^{(2)} = -\frac{i}{\omega\mu\rho}\frac{\partial^2\psi_2}{\partial\varphi\partial z}$$

$$E_z^{(2)} = 0, \qquad H_z^{(2)} = -\frac{i}{\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_2 \tag{4.4}$$

The fields $\underset{\sim}{E}^{(1)}$, $\underset{\sim}{H}^{(1)}$ are the transverse (to the z-axis) magnetic fields, while the fields $\underset{\sim}{E}^{(2)}$, $\underset{\sim}{H}^{(2)}$ are the transverse electric fields.

The most general solution for the potentials in terms of cylinderical coordinates is

$$\psi_i(z,\rho,\varphi) = \sum_{n=0}^{\infty} \int d\beta \; C_n^{(i)} \, e^{i\beta z} \cos n\varphi \; J_n[(k^2-\beta^2)^{1/2}\rho]$$

$$+ \sum_{n=1}^{\infty} \int d\beta \, D_n^{(i)} \, e^{i\beta z} \sin n\varphi \, J_n[(k^2-\beta^2)^{1/2}\rho]$$

$$+ \sum_{n=0}^{\infty} \int d\beta \, E_n^{(i)} \, e^{i\beta z} \cos n\varphi \, N_n[(k^2-\beta^2)^{1/2}\rho]$$

$$+ \sum_{n=1}^{\infty} \int d\beta \, F_n^{(i)} \, e^{i\beta z} \sin n\varphi \, N_n[(k^2-\beta^2)^{1/2}\rho] \ . \qquad (4.5)$$

In (4.5) the coefficients $C_n^{(i)}$, $D_n^{(i)}$, $E_n^{(i)}$ and $F_n^{(i)}$ are functions of β. For the moment we do not give the range of integration for the variable β. The functions J_n and N_n are the usual Bessel functions.

Let $E_r(r,\theta,\varphi)$ and $H_r(r,\theta,\varphi)$ be the radial components of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ as expressed in terms of the usual spherical coordinates. Using the techniques of Chapter 6 of Reference 3 one can show that for $\pi \geq \theta \geq \frac{\pi}{2}$ the φ-dependence of E_r and H_r is identical to that for E_r and H_r for the scattering of the plane wave (2.2) and (2.5) by a sphere. In particular, for $\theta = \frac{\pi}{2}$ we have

$$E_r = G(r) \cos \varphi \ ,$$

$$H_r = H(r) \sin \varphi \ ,$$

where H(r) and G(r) are functions of the radius only.

We shall require that at z=0, the components E_ρ and H_ρ obtained from (4.3) and (4.4) equal the components E_r and H_r given by the equation on the preceding page. This condition is a necessary condition for the solution of Maxwell's equations for the problem. We find that all the coefficients in (4.5) must vanish except $E_1^{(1)}$, $C_1^{(1)}$, $D_1^{(1)}$, and $F_1^{(1)}$. Our expressions for the potentials thus simplify considerably. On renaming the non-vanishing coefficients we obtain the following expressions for the potentials:

$$\psi_1 = \cos\varphi \int d\beta\, e^{i\beta z} \{A_1(\beta)\, J_1[(k^2-\beta^2)^{1/2}\rho] + B_1(\beta)\, N_1[(k^2-\beta^2)^{1/2}\rho]\}$$

$$\psi_2 = \sin\varphi \int d\beta\, e^{i\beta z} \{A_2(\beta)\, J_1[(k^2-\beta^2)^{1/2}\rho] + B_2(\beta)\, N_1[(k^2-\beta^2)^{1/2}\rho]\} \tag{4.6}$$

Expressions (4.6) for the potentials still provide rigorous and very general solutions of Maxwell's Equations for the cylindrical part of the scatterer. We shall now make further specializations which lead to reasonable fields $\underset{\sim}{E}$ and $\underset{\sim}{H}$ for $z > 0$. First of all, in order to preclude fields which increase or decrease exponentially with ρ--these fields being incompatible with a solution for a scattering problem--we restrict β to the real values

$$-k < \beta < +k \quad . \tag{4.7}$$

Secondly, because we expect the electromagnetic waves to move such that their propagation vector has a positive z-component, we exclude positive values of β. Hence

$$-k < \beta < 0 \quad . \tag{4.7a}$$

We shall now use the conditions that the tangential component of $\underset{\sim}{E}$ and the normal component of $\underset{\sim}{H}$ be zero on the surface of the cylinder. Thus

$$E_z(z,a,\varphi) = 0 \ ,$$

$$H_\rho(z,a,\varphi) = 0 \ ,$$

$$E_\varphi(z,a,\varphi) = 0 \ , \tag{4.8}$$

for $z > 0$.

On using (4.3), (4.4) and (4.6) the first of Equations (4.8) yields

$$\int_{-k}^{0} d\beta \ (k^2-\beta^2) e^{i\beta z} \ \{A_1(\beta) J_1[(k^2-\beta^2)^{1/2} a] +$$

$$+ B_1(\beta) N_1[(k^2-\beta^2)^{1/2} a]\} = 0 \tag{4.9}$$

for $z > 0$. Similarly the second of Equations (4.8) gives

$$\int_{-k}^{0} d\beta\ \beta(k^2-\beta^2)^{1/2}\{A_2(\beta)J_1'[(k^2-\beta^2)^{1/2}a]$$

$$+ B_2(\beta)N_1'[(k^2-\beta^2)^{1/2}a]\} = 0 \quad , \tag{4.10}$$

for $z > 0$. In (4.10) the prime means that a derivative is taken with respect to the argument. Equations (4.9) and (4.10) are satisfied if

$$\frac{A_1(\beta)}{B_1(\beta)} = -\frac{N_1[(k^2-\beta^2)^{1/2}a]}{J_1[(k^2-\beta^2)^{1/2}a]} \quad ,$$

$$\frac{A_2(\beta)}{B_2(\beta)} = -\frac{N_1'[(k^2-\beta^2)^{1/2}a]}{J_1'[(k^2-\beta^2)^{1/2}a]} \quad . \tag{4.11}$$

It is readily shown that the third of Equations (4.8) is also satisfied if Equations (4.11) hold.

We shall now use the fact that ka is small to simplify (4.11). We shall also use the fact that for calculations of surface current we need consider only in (4.6) only values of ρ near the surface of the cylinder so that $k\rho$ is also small.

Then since for small x

$$J_1(x) \cong \frac{x}{2} \quad ,$$

$$N_1(x) \cong -\frac{1}{\pi}\frac{2}{x} ,$$

$$J_1'(x) \cong \frac{1}{2}$$

$$N_1'(x) \cong \frac{1}{\pi}\frac{2}{x^2} , \tag{4.12}$$

we have for small ka and kρ

$$\psi_1 = \frac{2}{\pi}\left(\frac{\rho}{a^2} - \frac{1}{\rho}\right)\cos\varphi \int_{-k}^{0} d\beta \; e^{i\beta z} (k^2-\beta^2)^{-1/2} B_1(\beta) ,$$

$$\psi_2 = -\frac{2}{\pi}\left(\frac{\rho}{a^2} + \frac{1}{\rho}\right)\sin\varphi \int_{-k}^{0} d\beta \; e^{i\beta z} (k^2-\beta^2)^{-1/2} B_2(\beta) . \tag{4.13}$$

We must now make some assumptions about the coefficients $B_i(\beta)$. It seems reasonable to assume that the potentials (4.13) should yield electromagnetic fields which are smooth bundles of plane wave solutions. That is, it seems reasonable that no value of β should be too highly preferred.

Hence we shall assume that the functions $B_i(\beta)(k^2-\beta^2)^{-1/2}$ are smooth functions of β in the domain of integration and in fact we shall assume for convenience that these functions are constant. We can then carry out the integration over β in the integrals. We obtain

$$\psi_1 = D_1 \left(\frac{\rho}{a^2} - \frac{1}{\rho} \right) \cos \varphi \, \frac{1 - e^{-ikz}}{z} \quad ,$$

$$\psi_2 = D_2 \left(\frac{\rho}{a^2} + \frac{1}{\rho} \right) \sin \varphi \, \frac{1 - e^{-ikz}}{z} \quad , \tag{4.14}$$

where D_1 and D_2 are constants.

We shall now calculate the components of the surface currents on the cylinder. We shall use the requirement that the current (3.5) on the hemispherical cap must equal the current on the cylinder at the junction of the cap with the cylinder. From $\underset{\sim}{J} = \underset{\sim}{n} \times \underset{\sim}{H}$ where $\underset{\sim}{n}$ and $\underset{\sim}{H}$ are the normal to the surface of the cylinder and the magnetic field at the surface of the cylinder respectively, we have for the components of the surface current

$$J_\varphi = -H_z \quad ,$$

$$J_z = H_\varphi \quad , \tag{4.15}$$

where the components of $\underset{\sim}{H}$ are obtained from ψ_1, ψ_2 through (4.3), (4.4). Thus

$$J_\varphi = \sin \varphi \, \frac{i}{\omega\mu} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi_2 \tag{4.16}$$

From our requirement on the continuity of the current we require J_φ of Equations (3.5) equal J_φ of (4.16) when $\theta = \frac{\pi}{2}$ or, equivalently,

when $z = 0$. We see from (3.5) that $J_\varphi = 0$ at the junction of the spherical cap with the cylinder. Hence

$$D_2 = 0 \quad , \tag{4.17}$$

from which it follows that

$$\psi_2 = 0 \quad . \tag{4.18}$$

From (4.15), (4.3), (4.4), (4.14) and (4.18)

$$J_z = - \left. \frac{\partial \psi_1}{\partial \rho} \right|_{\rho = a} ,$$

$$= - D_1 \cos \varphi \frac{1-e^{-ikz}}{z} \quad . \tag{4.19}$$

In (4.19) we have absorbed a factor of $(2/a^2)$ into D_1.

We shall now require that J_θ of (3.5) equal $-J_z$ of (4.19) when $\theta = \frac{\pi}{2}$ and $z = 0$. We find for D_1 the result

$$D_1 = -i \frac{3}{2} \frac{E_o}{k\eta} \quad . \tag{4.20}$$

Thus we have found the surface current on the cylinder. We now summarize our result for the current taking, however, the complex conjugate so that we will now work with the time factor $e^{-i\omega t}$ as in the evaluation of $\underset{\sim}{I}_1$.

$$J_z = -i\,\frac{3}{2}\,\frac{E_o}{\eta}\,\frac{1 - e^{ikz}}{kz}\ ,$$

$$J_\varphi = 0\ . \tag{4.21}$$

We shall now evaluate $\underset{\sim}{I}_2$. In (2.8a) we use

$$\underset{\sim}{J}(\underset{\sim}{r}') = J_z(z')\underset{\sim}{i}_z$$

$$= J_z(z')[\hat{\underset{\sim}{r}} \cos\theta - \hat{\underset{\sim}{\theta}} \sin\theta] \tag{4.22}$$

where the unit vector $\hat{\underset{\sim}{r}}$ and $\hat{\underset{\sim}{\theta}}$ are the same as those of the previous sections and $J_z(z)$ is given by the first of Equations (4.21). It is easily seen that

$$\hat{\underset{\sim}{r}} \times \hat{\underset{\sim}{r}} \times \underset{\sim}{J}(\underset{\sim}{r}') = \sin\theta\, J_z(z')\,\hat{\underset{\sim}{\theta}}\ . \tag{4.23}$$

Also

$$\underset{\sim}{r}'\cdot\hat{\underset{\sim}{r}} = a \sin\theta \cos(\varphi - \varphi') + z' \cos\theta \tag{4.24}$$

where φ' and z' are the cylindrical coordinates on the surface of the cylinder contained in $\underset{\sim}{r}'$

Thus

$$\underset{\sim}{I}_2 = -i\,\frac{3}{2}\,\frac{E_o}{k\eta}\,\sin\theta\; a\; \hat{\underset{\sim}{\theta}} \int_0^{2\pi} d\varphi' \cos\varphi'\; e^{-ika\sin\theta\cos(\varphi-\varphi')} \times$$

$$\int_0^\infty dz\, e^{-ikz} \cos\theta\, \frac{1 - e^{ikz}}{z} \quad . \tag{4.25}$$

Let us first consider the integral over φ'. Because of the smallness of ka we expand the exponent in powers of ka. The 0'th power yields nothing when integrated. Hence we must use the first power to evaluate our integral to the lowest non-vanishing order. For small ka we then obtain

$$\int_0^{2\pi} d\varphi' \cos\varphi'\, e^{-ika \sin\theta \cos(\varphi-\varphi')} \simeq -i\pi ka \sin\theta \cos\varphi \tag{4.26}$$

To evaluate the second integral in (4.25) we use the theorem

$$\int_0^\infty \frac{e^{iaz} - e^{ibz}}{z}\, dz = 0\ , \quad \text{if the sign of a equals the sign of b}$$

$$= -\pi i \operatorname{sgn} b, \text{ if the sign of a is the negative of the sign of b.} \tag{4.27}$$

We thus obtain as our answer for $\underset{\sim}{I_2}$ the following expression

$$\underset{\sim}{I_2} = \underset{\sim}{\theta}\{- \frac{3}{2} (\pi a)^2 \frac{E_o}{\eta} \sin^2\theta \cos\varphi\, A(\theta)\} \tag{4.28}$$

where $A(\theta)$ is defined by (1.3a). It is clear that for back scattering, i.e. for $\pi > \theta > \frac{\pi}{2}$, the current on the cylinder gives no contribution.

Having evaluated $\underset{\sim}{I}_1$ and $\underset{\sim}{I}_2$ we can carry out the procedure outlined in Section 2 to obtain the cross section which we desire.

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